Decoupling Techniques for Coupled PDEs

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OUTLINE

- Driving Forces
- Case-study: Coupling Fluid Flow with Porous Media Flow
- Decoupling Techniques

  1. Two-grid Decoupling and Linearization
  2. Domain Decomposition and Decoupled Preconditioners
  3. Temporal Decoupling and Linearization

DRIVING FORCES

Coupled PDE Models

1. Local models (heterogeneous or homogeneous) coupled through interface conditions in multi-physics applications such as coupled fluid/porous-media flows, fluid/solid interaction, etc. or in parallel/network computing
2. Multiple coupled PDEs with various physical variables, such as the GL model in superconductivity
3. Multiple coupled components with various physical or computational properties, such as linear vs nonlinear, symmetric vs non-symmetric, or of multiple scales

Decoupling

1. Software reuse and integration
2. Grid/Network Computing: distributed resource-sharing (both hardware and software) with component-based services available on the network; collaborative research and simulation,

Abstract Coupled Model for Decoupling Techniques: Algorithms and Numerical Analysis

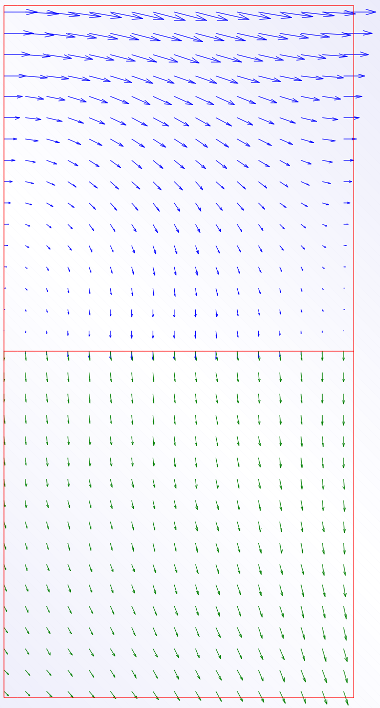
An Abstract Coupled Model: Operator Splitting, Correction, and Decoupling

Find $u \in U$, such that $Au \equiv (D + C)u = f$

1. The solution space U may be a cross product of a set of subspaces for the associated physical variables defined on subregions or the same domain, depending on the type of the coupling
2. $Dv = g$: decoupled, easily solvable by existing efficient solvers, etc.
3. The coupling operator C is relatively weak and dominated by the leading operator D.

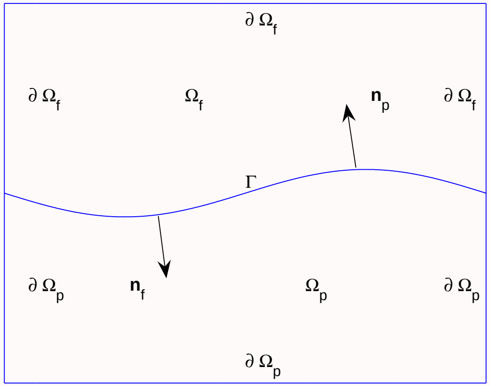
A CASE-STUDY

Filtration Process in Porous Media Applications



Fluid flow coupled with porous media flow

Mathematical (Macroscopic) Modeling



Fluid flow coupled with porous media flow

Governing Equations of the Fluid Flow

- Stokes equations (time-dependent, linear case): $\forall t > 0$, $\forall \mathbf{x} \in \Omega_f$

$$\begin{cases} \frac{\partial \mathbf{u}_f}{\partial t} - \mathbf{divT}(\mathbf{u}_f, p_f) \equiv \frac{\partial \mathbf{u}_f}{\partial t} - \nu \triangle \mathbf{u}_f + \triangledown p = \mathbf{g_f}, & \text{(conservation of momentum)} \\ -\mathbf{div}\mathbf{u}_f = 0, & \text{(conservation of mass)} \end{cases} \tag{1}$$

  - $\mathbf{T}(\mathbf{u}_f, p_f) = -p_f\mathbf{I} + 2\nu\mathbf{D}(\mathbf{u}_f)$: Stress tensor
  - $\mathbf{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla \mathbf{u}_f + \nabla^T \mathbf{u}_f)$: Deformation rate tensor
  - ν and $\mathbf{g_f}$: kinematic viscosity and external force

Governing Equations of the Porous Media Flow

- Darcy's law: $\forall t > 0, \forall \mathbf{x} \in \Omega_p$

$$\begin{cases} S_0 \frac{\partial \phi}{\partial t} + n \mathbf{div} \mathbf{u}_p = g_p, & \text{(conservation of mass)} \\ \mathbf{u}_p = -\frac{\mathbf{K}}{n} \nabla \phi, & \text{(Darcy's law)} \end{cases} \tag{2}$$

  - S_0 and n: mass storativity and volumetric porosity
  - source g_p satisfies the solvability condition $\int_{\Omega_p} g_p = 0$
  - $\mathbf{K} = diag\{K_{11}, \cdot, K_{dd}\}$: hydraulic conductivity tensor of the porous medium with $K_{i,i} \propto \frac{\epsilon^2}{\nu}$ being positive
  - $\phi = z + \frac{p_p}{\rho_f g}$: piezometric head
  - z: elevation from a reference level (assumed to be 0)
  - p_p, ρ_f, g: pressure, density, and gravity acceleration.

Interface Coupling Conditions

- Continuity of normal velocities

$$\mathbf{u}_f \cdot \mathbf{n}_f = \mathbf{u}_p \cdot \mathbf{n}_f = -\frac{\mathbf{K}}{n}\frac{\partial \phi}{\partial \mathbf{n}_f}$$

- Balance of normal forces

$$-[(\mathbf{T}(\mathbf{u}_f, p_f)) \cdot \mathbf{n}_f] \cdot \mathbf{n}_f = \rho_f g \phi$$

- BJS Condition (Beavers, Joesph 1967, Saffman 1971, Jager, Miliek 2000): the slip velocity along Γ is proportional to the shear stress along Γ

$$-[(\mathbf{T}(\mathbf{u}_f, p_f)) \cdot \mathbf{n}_f] \cdot \tau_i = \frac{\alpha}{\sqrt{\tau_i \cdot \mathbf{K} \cdot \tau_i}}(\mathbf{u}_f - \mathbf{u}_p) \cdot \tau_i \approx \frac{\alpha}{\sqrt{\tau_i \cdot \mathbf{K} \cdot \tau_i}}\mathbf{u}_f \cdot \tau_i,\ i = 1, ..., d-1$$

  - $\{\tau_i\}_{i=1}^{d-1}$: unit tangential vectors, where d is the spacial dimension
  - $\alpha > 0$: experimentally determined, depending on the properties of the porous medium

Weak Formulation

(Time-dependent, Linear Case)

- Find $u = (\mathbf{u}, \phi) \in W, \; p \in Q$, such that

$$\begin{cases} \left(\frac{\partial u}{\partial t}, v\right) + a(u,v) + b(v,p_f) = (f,v), \;\; \forall v = (\mathbf{v}, \psi) \in W, \\ \\ b(u,q) = 0, \;\; \forall q \in Q. \end{cases} \qquad (3)$$

  - $W = H_f \times H_p$,
  - $Q = L^2(\Omega_f)$,
  - $H_p = \{\phi \in H^1(\Omega_p) \mid \phi = 0 \; on \; \partial\Omega_{p,D}\}$,
  - $H_f = \{\mathbf{v} \in (H^1(\Omega_f))^d \mid \mathbf{v} = 0 \; on \; \partial\Omega_{f,D}\}$

Weak Formulation (Continued)

(Time-dependent, Linear Case)

- $(u,v) = (\mathbf{u},\mathbf{v})_{\Omega_f} + (\varphi,\psi)_{\Omega_p} = n\int_{\Omega_f}\mathbf{u}\cdot\mathbf{v} + \rho_f g S_0\int_{\Omega_p}\varphi\psi,$

- $b(v,p) = -\int_{\Omega_f} np\,\mathrm{div}\mathbf{v},$

- $a(u,v) = a_f(\mathbf{u},\mathbf{v}) + a_p(\phi,\psi) + a_\Gamma(u,v) \equiv a_\Omega(u,v) + a_\Gamma(u,v),$

  - $a_f(\mathbf{u},\mathbf{v}) = \int_{\Omega_f} 2n\nu D(\mathbf{u})\cdot D(\mathbf{v}) + \sum_{i=1}^{d-1}\frac{\alpha n}{\sqrt{\tau_i\cdot\mathbf{K}\cdot\tau_i}}\int_\Gamma(\mathbf{u}\cdot\tau_i)(\mathbf{v}\cdot\tau_i)$

  - $a_p(\phi,\psi) = \int_{\Omega_p}\rho_f g\nabla\psi\cdot\mathbf{K}\nabla\phi$

  - $a_\Gamma(u,v) = \int_\Gamma n\rho_f g[\phi\mathbf{v} - \psi\mathbf{u}]\cdot\mathbf{n}_f$

Weak Formulation

(Time-dependent, Nonlinear Case)

- Find $u = (\mathbf{u}, \phi) \in W,\ p \in Q$, such that

$$\begin{cases} \left(\frac{\partial u}{\partial t}, v\right) + a(u; u, v) + b(v, p_f) = (f, v), \ \ \forall v = (\mathbf{v}, \psi) \in W, \\ \\ b(u, q) = 0, \ \ \forall q \in Q. \end{cases} \tag{4}$$

  - $W = H_f \times H_p$,
  - $Q = L^2(\Omega_f)$,
  - $H_p = \{\phi \in H^1(\Omega_p) \mid \phi = 0 \ on \ \partial\Omega_{p,D}\}$,
  - $H_f = \{\mathbf{v} \in (H^1(\Omega_f))^d \mid \mathbf{v} = 0 \ on \ \partial\Omega_{f,D}\}$

Weak Formulation (Continued)

(Time-dependent, Nonlinear Case)

- $a(w; u, v) = a_f(\mathbf{u}, \mathbf{v}) + a_{f,c}(\mathbf{w}; \mathbf{u}, \mathbf{v}) + a_p(\phi, \psi) + a_\Gamma(u, v)$
  - Convection term

$$a_{f,c}(\mathbf{w}; \mathbf{u}, \mathbf{v}) = n\rho_f \frac{1}{2}[\int_{\Omega_f} (\mathbf{w} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} - \int_{\Omega_f} (\mathbf{w} \cdot \nabla)\mathbf{v} \cdot \mathbf{u} + \int_\Gamma (\mathbf{w} \cdot \mathbf{n}_f)(\mathbf{u} \cdot \mathbf{v})]$$

  - An inertial energy term $a_{\Gamma,e}(\mathbf{w}; \mathbf{u}, \mathbf{v})$ on the interface might be included (physical???)

$$a_{\Gamma,e}(\mathbf{w}; \mathbf{u}, \mathbf{v}) = -n\rho_f \frac{1}{2} \int_\Gamma (\mathbf{w} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{n}_f)$$

DECOUPLING TECHNIQUES

TWO-GRID DECOUPLING AND LINEARIZATION

TWO-GRID DECOUPLING

The Steady-state Linear Setting (M. Mu and J.C. Xu)

1. Solve a coarse-grid problem with spacing H: find $u_H = (\mathbf{u_H}, \phi_H) \in W_H \subset W_h,\ p_H \in Q_H \subset Q_h$ such that

$$\begin{cases} a(u_H, v_H) + b(v_H, p_H) = f(v_H), & \forall v_H = (\mathbf{v_H}, \psi_H) \in W_H, \\ b(u_H, q_H) = 0, & \forall q_H \in Q_H; \end{cases} \tag{5}$$

2. Solve a modified fine-grid problem: find $u^h = (\mathbf{u^h}, \phi^h) \in W_h,\ p^h \in Q_h$ such that

$$\begin{cases} a_\Omega(u^h, v_h) + b(v_h, p^h) = f(v_h) - a_\Gamma(u_H, v_h), & \forall v_h \in W_h, \\ b(u^h, q_h) = 0, & \forall q_h \in Q_h. \end{cases} \tag{6}$$

DECOUPLED FINE-GRID PROBLEM

The fine-grid problem is equivalent to the decoupled subproblems:

- Discrete Stokes problem: find $\mathbf{u^h} \in H_{f,h},\ p^h \in Q_h$ such that

$$\begin{cases} a_{\Omega_f}(\mathbf{u}^h, \mathbf{v}_h) + b(\mathbf{v}_h, p^h) = (n\mathbf{g}_f, \mathbf{v}_h) - \int_\Gamma n\rho_f g \phi_H \mathbf{v}_h \cdot \mathbf{n}_f, & \forall \mathbf{v}_h \in H_{f,h}, \\ b(\mathbf{u}^h, q_h) = 0, & \forall q_h \in Q_h. \end{cases} \tag{7}$$

- Discrete Darcy problem: find $\phi^h \in H_{p,h}$ such that

$$a_{\Omega_p}(\phi^h, \psi_h) = (\rho_f g_p, \psi_h) + \int_\Gamma n\rho_f g \psi_h \mathbf{u}_H \cdot \mathbf{n}_f, \forall \psi_h \in H_{p,h}. \tag{8}$$

DIFFERENTIAL FORM OF THE FINE-GRID PROBLEM

- Discrete Stokes problem:

$$\begin{cases} -\nu\triangle\mathbf{u}_f + \bigtriangledown p_f = \mathbf{f}, & \forall\mathbf{x}\in\Omega_f, \\ -\mathrm{div}\mathbf{u} = 0, & \forall\mathbf{x}\in\Omega_f, \end{cases} \tag{9}$$

with the boundary conditions:

$$\begin{cases} \mathbf{u}_f = \mathbf{u}_D, & \text{on } \partial\Omega_f/\Gamma, \\ p_f - 2\nu\mathbf{n}_f\cdot\mathbf{D}(\mathbf{u}_f)\cdot\mathbf{n}_f = \rho g\phi_H, & \text{on } \Gamma, \\ -\frac{\sqrt{\tau_i\cdot\mathbf{K}\cdot\tau_i}}{\alpha_{BJ}}2\mathbf{n}_f\cdot\mathbf{D}(\mathbf{u}_f)\cdot\tau_i = \mathbf{u}_{f,H}\cdot\tau_i,\ i=1,...,d-1, & \text{on } \Gamma. \end{cases} \tag{10}$$

- Discrete Darcy problem:

$$-\mathrm{div}\mathbf{K}\nabla\phi = 0, \quad \forall\mathbf{x}\in\Omega_p. \tag{11}$$

with the boundary conditions:

$$\begin{cases} \phi = \phi_D, & \text{on } \partial\Omega_p/\Gamma, \\ \mathbf{K}\nabla\phi\cdot\mathbf{n}_p = \mathbf{u}_{f,H}\cdot\mathbf{n}_f, & \text{on } \Gamma. \end{cases} \tag{12}$$

THEORY OF THE TWO-GRID ALGORITHM

- Errors induced from the coarse-grid decoupling:

$$||u_h - u^h||_W \lesssim H^2,$$

$$||p_h - p^h||_Q \lesssim H^2.$$

- Error estimates of the two-grid method with $H = \sqrt{h}$:

$$||u - u^h||_W + ||p - p^h||_Q \leq (||u - u_h||_W + ||u_h - u^h||_W) + (||p - p_h||_Q + ||p_h - p^h||_Q) \lesssim h + H^2 \lesssim h,$$

TWO-GRID DECOUPLING AND LINEARIZATION

The Steady-state Nonlinear Setting (M. Mu, J.C. Xu and M.C. Cai)

Coarse-grid approximation: $a_{f,n}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + a_\Gamma(u_h, v_h) \approx$

$$[a_{f,n}(\mathbf{u}_H; \mathbf{u}^h, \mathbf{v}_h) + a_{f,n}(\mathbf{u}^h; \mathbf{u}_H, \mathbf{v}_h) - a_{f,n}(\mathbf{u}_H; \mathbf{u}_H, \mathbf{v}_h)] + a_\Gamma(u_H, v_h)$$

1. Solve a coarse-grid problem with spacing H: find $u_H = (\mathbf{u}_H, \phi_H) \in W_H \subset W_h$, $p_H \in Q_H \subset Q_h$ such that

$$\begin{cases} a(u_H; u_H, v_H) + b(v_H, p_H) = (f, v_H) & \forall v_H = (\mathbf{v}_H, \psi_H) \in W_H \\ b(u_H, q_H) = 0 & \forall q_H \in Q_H; \end{cases} \tag{13}$$

TWO-GRID DECOUPLING AND LINEARIZATION (Continued)

Solve two decoupled and linear local problems on a fine grid:

1. Find $\phi^h \in H_{p,h}$ such that

$$a_p(\phi^h, \psi_h) = \rho_f g \int_{\Omega_p} f_p \psi_h + n\rho_f g \int_\Gamma \psi_h \mathbf{u}_H \cdot \mathbf{n}_f \; \forall \psi_h \in H_{p,h}; \tag{14}$$

2. Find $(\mathbf{u}^h, p^h) \in H_{f,h} \times Q_h$ such that

$$\begin{cases} \tilde{a}_f(\mathbf{u}_H; \mathbf{u}^h, \mathbf{v}_h) + b(\mathbf{v}_h, p^h) = (\tilde{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in H_{f,h} \\ b(\mathbf{u}^h, q_h) = 0 & \forall q_h \in Q_h, \end{cases} \tag{15}$$

where $\tilde{a}_f(\mathbf{w}; \mathbf{u}, \mathbf{v}) = a_f(\mathbf{u}, \mathbf{v}) + a_{f,n}(\mathbf{w}; \mathbf{u}, \mathbf{v}) + a_{f,n}(\mathbf{u}; \mathbf{w}, \mathbf{v})$ and

$(\tilde{f}, \mathbf{v}_h) = n \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}_h + a_{f,n}(\mathbf{u}_H; \mathbf{u}_H, \mathbf{v}_h) - n\rho_f g \int_\Gamma \phi_H \mathbf{v}_h \cdot \mathbf{n}_f$.

THEORY OF THE TWO-GRID ALGORITHM FOR THE COUPLED NS/DARCY MODEL

- L^2- error estimate for the coupled and nonlinear algorithm
- Errors induced from the coarse-grid decoupling:

$$||u_h - u^h||_W \lesssim H^2,$$

$$||p_h - p^h||_Q \lesssim H^2.$$

- Error estimates of the two-grid method with $H = \sqrt{h}$:

$$||u - u^h||_W + ||p - p^h||_Q \lesssim h,$$

DECOUPLED PRECONDITIONERS

Preconditioned GMRES For The Stokes/Darcy Model

- Discrete problem by the mixed finite element method

$$\begin{bmatrix} A_{pp} & A_{p\Gamma} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{\Gamma p} & A^{p}_{\Gamma\Gamma} & M^{T}_{\Gamma\Gamma} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -M_{\Gamma\Gamma} & A^{f}_{\Gamma\Gamma} & A^{T}_{f\Gamma} & B^{T}_{\Gamma} \\ \mathbf{0} & \mathbf{0} & A_{f\Gamma} & A_{ff} & B^{T}_{f} \\ \mathbf{0} & \mathbf{0} & B_{\Gamma} & B_{f} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \phi_{int} \\ \phi_{\Gamma} \\ \mathbf{u}_{\Gamma} \\ \mathbf{u}_{int} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{p} \\ \mathbf{f}_{p\Gamma} \\ \mathbf{f}_{pf} \\ \mathbf{f}_{f\Gamma} \\ \mathbf{f}_{f} \end{bmatrix}$$

- Saddle-point notation in $W \times Q$

$$M \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} \equiv \begin{bmatrix} A & B^{T} \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- A is coupled and nonsymmetric

$$A = \begin{bmatrix} A_{p} & A^{T}_{\Gamma} \\ -A_{\Gamma} & A_{f} \end{bmatrix}$$

Preconditioning for the Saddle-point Problem

- Schur complement based precondtioners (Golub 2000, Ipsen 2001)

$$P_{+} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & BA^{-1}B^T \end{bmatrix}, \quad \text{or } P_T = \begin{bmatrix} A & \mathbf{0} \\ B & BA^{-1}B^T \end{bmatrix}$$

- $H = P_{+}^{-1}M$ satisfies a polynomial equation of degree 4, $H = P_{T}^{-1}M$ satisfies a polynomial equation of degree 2

- Preconditioned Krylov subspace iteration converges in 4 iterations

  The number of iterations of the Krylov subspace method is determined by $dim(\mathbf{r}_0, H\mathbf{r}_0, H^2\mathbf{r}_0, H^3\mathbf{r}_0, ..., H^n\mathbf{r}_0)$

Decoupled Preconditioners for the Stokes/Darcy Model

(M. Mu, J.C. Xu, and M.C.Cai)

- Block diagonal preconditioner

$$P_{\pm} = \begin{bmatrix} A_0 & 0 \\ 0 & \pm J_0 \end{bmatrix},$$

  - $A_0 = \begin{bmatrix} A_p & 0 \\ 0 & A_f \end{bmatrix}$ is spectrally equivalent approximation to the stiffness matrices A
  - J_0 is the mass pressure matrix, and is spectrally equivalent to the Schur complement $BA_0^{-1}B^T$ (Verfurth 1984)

- Block triangular preconditioner

$$P_{T_1} = \begin{bmatrix} A_0 & 0 \\ B & -J_0 \end{bmatrix}$$

- Block triangular preconditioner with scaling factors

$$P_T(k,\rho) = \begin{pmatrix} kA_0 & 0 \\ B & -\rho J_0 \end{pmatrix}$$

- Replacing A_0 by a better approximation

$$P_{T_2}(\rho) = \begin{pmatrix} A_p & 0 & 0 \\ -A_\Gamma & A_f & 0 \\ 0 & B_f & -\frac{1}{\rho}J_0 \end{pmatrix}$$

Convergence Analysis for $P_T(k,\rho)$

- FOV-equivalence $M \approx_H P$: Assume H is block diagonal and SPD. Nonsingular matrices $M, P \in R^{n\times n}$ are said to be $H-$ FOV equivalent, if

$$\gamma \le \frac{(x, P^{-1}Mx)_H}{(x,x)_H}, \quad \frac{||P^{-1}Mx||_H}{||x||_H} \le \Gamma, \forall x \in R^n \setminus \mathbf{0},$$

  where γ and Γ are independent of n.

- Convergence of the preconditioned GMRES: If $P \approx_H M$, then the preconditioned GMRES iteration converges with

$$\frac{||\mathbf{r}^k||_H}{||\mathbf{r}_0||_H} \le (1 - \frac{\gamma^2}{\Gamma^2})^{k/2}$$

- **Theorem** There exist k and ρ such that

$$\gamma = \inf_{\forall \mathbf{0}\neq\mathbf{x}} \frac{\left[\mathbf{x}, P_T(k,\rho)^{-1}M\mathbf{x}\right]_{P_+}}{[\mathbf{x},\mathbf{x}]_{P_+}} > 0 \quad \text{and} \quad \Gamma = \sup_{\forall \mathbf{0}\neq\mathbf{x}} \frac{||P_T(k,\rho)^{-1}M\mathbf{x}||_{P_+}}{||\mathbf{x}||_{P_+}}$$

  are independent of h. It follows that the P_+-norm based GMRES algorithm, applied to $P_T(k,\rho)^{-1}M\mathbf{x} = P_T(k,\rho)^{-1}\mathbf{b}$, converges independent of h. Moreover, the residuals satisfy

$$\frac{||P_T(k,\rho)^{-1}\mathbf{r}_q||_{P_+}}{||P_T(k,\rho)^{-1}\mathbf{r}_0||_{P_+}} \le (1 - \frac{\gamma^2}{\Gamma^2})^{q/2}.$$

TEMPORAL EXTRAPOLATION

Coupled Marching Algorithms

- Coupled Backward Euler Scheme (CBES): Find $u_h^m = (\mathbf{u}_{fh}^m, \varphi_h^m) \in W_h$ and $p_{fh}^m \in Q_h$, $m = 1, 2, \cdots, J$, such that $\forall v_h = (\mathbf{v}_h, \psi_h) \in W_h$ and $\forall q_h \in Q_h$

$$\begin{cases} \left(\frac{u_h^m - u_h^{m-1}}{k}, v_h\right) + a(u_h^m, v_h) + b(v_h, p_{fh}^m) = (f^m, v_h), \\ b(u_h^m, q_h) = 0, \\ u_h^0 = R_h u^0. \end{cases} \tag{16}$$

  - $W_h = H_{f,h} \times H_{p,h} \subset W$ and $Q_h \subset Q$ are two finite element spaces.
  - Discrete inf-sup condition on $(H_{f,h}, Q_h)$ (finite elements for the Stokes problem (Brezzi and Fortin)):

$$b(\mathbf{v_h}, q_h) \geq \beta^* ||\mathbf{v_h}||_{H_f} ||q_h||_Q$$

- Higher order marching schemes, such as the Crank-Nicolson scheme, may be applied, if necessary.

Decoupling

- A coupled problem must be solved at each time level.

- In principle, the decoupled methods developed for the stationary model can all be applied here at each time level. However, those methods either involve an iterative procedure, or a coarse-grid solver, or have other overhead and disadvantages.

Decoupled Marching Algorithms

- Decoupled Backward Euler Scheme (DBES): Find $u^{h,m} = (\mathbf{u}_f^{h,m}, \varphi^{h,m}) \in W_h$ and $p_f^{h,m} \in Q_h$, $m = 1, \cdots, J$, such that $\forall v_h = (\mathbf{v}_h, \psi_h) \in W_h$ and $\forall q_h \in Q_h$

$$\begin{cases} \left(\frac{u^{h,m}-u^{h,m-1}}{k}, v_h\right) + a_\Omega(u^{h,m}, v_h) + b(v_h, p_f^{h,m}) = (f^m, v_h) - a_\Gamma(u^{h,m-1}, v_h), \\ b(u^{h,m}, q_h) = 0, \\ u^{h,0} = R_h u^0. \end{cases} \tag{17}$$

- To improve the approximation accuracy, $a_\Gamma(u^{h,m-1}, v_h)$ may be replaced by a higher order extrapolation, say $a_\Gamma(2u^{h,m-1} - u^{h,m-2}, v_h)$, due to the second order extrapolation

$$u^m = 2u^{m-1} - u^{m-2} + O(k^2).$$

Stability

- Both CBES and DBES are unconditionally stable.
- Stability of DBES: Under the stability condition $\delta t = O(1)$, there hold

$$\begin{cases} n\|\mathbf{u}_f^{h,J}\|_0^2 + \rho_f g S_0 \|\varphi^{h,J}\|_0^2 + n\nu k \sum_{m=1}^J \|\nabla \mathbf{u}_f^{h,m}\|_0^2 + \rho_f g K k \sum_{m=1}^J \|\nabla \varphi^{h,m}\|_0^2 \le \kappa_0, \\ n\nu \|\nabla \mathbf{u}_f^{h,J}\|_0^2 + \rho_f g K \|\nabla \varphi^{h,J}\|_0^2 + nk \sum_{m=1}^J \|d_t \mathbf{u}_f^{h,m}\|_0^2 + \rho_f g S_0 k \sum_{m=1}^J \|d_t \varphi^{h,m}\|_0^2 \le \kappa_1, \\ n\|d_t \mathbf{u}_f^{h,J}\|_0^2 + \rho_f g S_0 \|d_t \varphi^{h,J}\|_0^2 + n\nu k \sum_{m=1}^J \|\nabla d_t \mathbf{u}_f^{h,m}\|_0^2 + \rho_f g K k \sum_{m=1}^J \|\nabla d_t \varphi^{h,m}\|_0^2 \le \kappa_2, \\ \|p_f^{h,J}\|_0^2 \le \kappa_3. \end{cases}$$

Error Estimates for DBES

- Error estimates are derived for the difference (e^m, η^m) between DBES and CBES

  where $e^m = (\mathbf{e}^m, \xi^m)$ with $\mathbf{e}^m = \mathbf{u}^m_{fh} - \mathbf{u}^{h,m}_f$ and $\xi^m = \varphi^m_h - \varphi^{h,m}$, and $\eta^m = p^m_{fh} - p^{h,m}_f$.

- Error estimates for DBES:

$$
\begin{cases}
\|u^{h,m} - u(t_m)\|_0 \lesssim \delta t + h^2, \\
\|\nabla(u^{h,m} - u(t_m))\|_0 \lesssim \delta t + h, \\
\|p_f^{h,m} - p_f(t_m)\|_0 \lesssim \delta t + h.
\end{cases}
$$

Conclusions

- Coupled PDE model: Find $u \in U$, such that $Au \equiv (D + C)u = f$

- Coupled discrete model: Find $u_h \in U_h$, such that $A_h u_h \equiv (D_h + C_h)u_h = f_h$

- Decoupled approaches

1. Approximation-based: Find $u^h \in U_h$, such that $D_h u^h = f_h - C^h \bar{u}_h$

   Then, $||u - u^h|| \lesssim ||u - u_h|| + ||u_h - u^h|| \lesssim ||u - u_h|| + ||D_h^{-1}(C_h u_h - C^h \bar{u}_h)||$

2. Preconditioning-based: $u_h^{k+1} = u_h^k + P_h(f_h - A_h u_h^k)$, with $P_h \approx A_h^{-1} \approx D_h^{-1}$